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VECTOR VALUED FUNCTION

4.1 Scalar and Vector Field Functions

A **vector functions** are functions whose values are **vectors**:

$$v = v(p) = (v_1(p), v_2(p), v_3(p))$$

depending on the points p in space, and **Scalar functions** are functions whose values are **scalars**:

$$f = f(p)$$

depending on p.

Definition 1 ***A scalar function*** *$F(x, y, z)$ defined over some region of space D is a function that assigns to each point P_0 in D with coordinates (x_0, y_0, z_0) the number $F(P_0) = F(x_0, y_0, z_0)$. The set of all numbers F(P) for all points P in D are said to form a* ***scalar field*** *over D.*

Definition 2

1. Let D be a set in $\Re^2$(a plane region). A vector field on $\Re^2$ is a function $F(x, y)$ that assigns to each point (x, y) in D a two-dimensional vector:

   $$F(x, y) = f(x, y)i + g(x, y)j$$

   where f and g are scalar functions.

2. Let D be a subset in $\Re^3$. **A vector field** on $\Re^3$ is a function $F(x, y, z)$ that assigns to each point (x, y, z) in D a three-dimensional vector:

   $$F(x, y, z) = f(x, y, z)i + g(x, y, z)j + h(x, y, z)k$$

   where f, g and h are scalar functions.

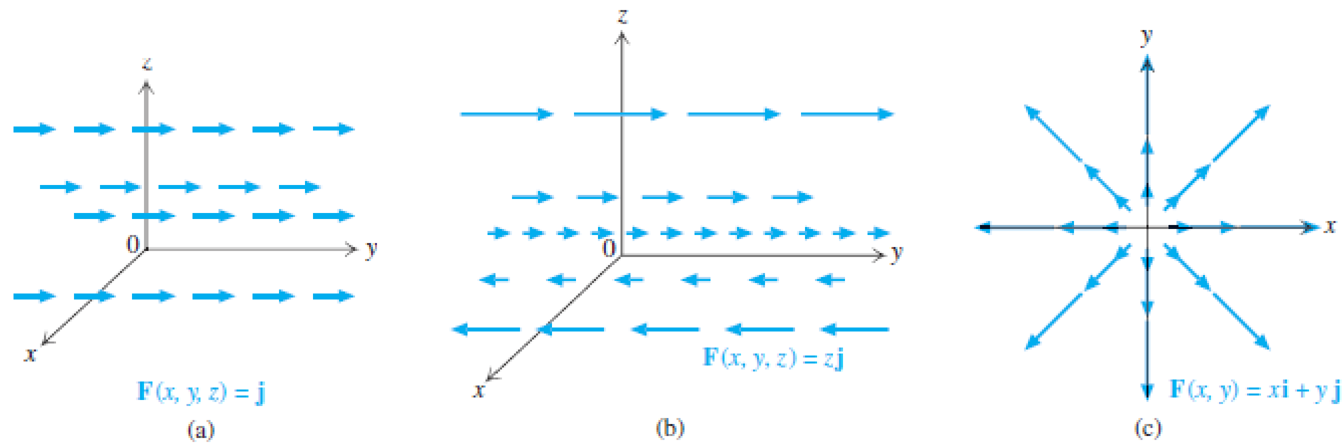


Figure 4.1: Examples of vector fields

Example

1. The scalar function of position $F(x, y, z) = xyz^2$ for (x, y, z) inside the unit sphere $x^2 + y^2 + z^2 = 1$ defines a scalar field throughout the unit sphere.

2. The distance $f(p)$ of any point p from a fixed point p_O in space is a scalar function. $f(P)$ defines a scalar field in space and given by
$$f(P) = f(x, y, z) = \sqrt{(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2}$$

3. The vector function $F(x, y, z) = (x - y)i + (y - z)j + (xyz - 2)k$ for (x, y, z) inside the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, defines a vector field throughout the ellipsoid.

4.2 Vector-valued functions

Definition 3 ***A Vector valued function*** *is simply a function whose domain is a set of real numbers and whose range is a set of vectors. The vector valued functions whose values are three-dimensional vectors is given by*
$$r(t) = (f(t), g(t), h(t)) = f(t)i + g(t)j + h(t)k$$
where the scalars $f(t)$, $g(t)$ and $h(t)$ are components of $r(t)$.

Example:If $r(t) = (t^3, ln(3 - t), \sqrt{t})$, then

1. the domain of $r(t)$ is $[0, 3)$
2. $r(0) = (0, ln3, 0) = ln3j$
3. $r(1) = (1, ln2, 1) = i + ln2j + k$

Example: The vector valued function $r(t) = costi + 2t^2j + 3tk$, then $r(0) = i$, $r(\pi) = -i + 2\pi^2j + 3\pi k$ and $r(2) = cos2i + 8j + 6k$
Note The vector valued function in $\Re^2$ has the form $r(t) = f(t)i + g(t)j$ where $f(t)$ and $g(t)$ are the component functions of r.

4.3 Calculus of vector valued functions

4.3.1 Limits and Continuity

The limit of a vector function is defined by taking the limits of its component functions as follows:

Definition 4 *If* $r(t) = (f(t), g(t), h(t))$*, then*

$$\lim_{t\to t_o} r(t) = (\lim_{t\to t_o} f(t), \lim_{t\to t_o} g(t), \lim_{t\to t_o} h(t))$$

provided the limits of the component functions exist.

Examples:

* If $r(t) = (1 + t^3)i + te^{-t}j + \frac{sint}{t}k$, then $\lim_{t\to 0} r(t) = i + k$

  If $r(t) = (2 + t^2)i + (t + 1)j + \frac{t}{t^2+1}k$, then $\lim_{t\to 1} r(t) = 3i + 2j + \frac{1}{2}k$

* **Definition 5** *A* vector valued function $r(t)$ *is* ***continuous*** *at* t_o *if*

$$\lim_{t\to t_o} r(t) = r(t_o)$$

A vector valued function is continuous for each value of t *if each component functions are continuous for all real number* t.

Example: Let $r(t) = \frac{1}{2-t}i + lntk$, then $r(t)$ is continuous for all $t > 0$ with $t \neq 2$.

Example: Let $r(t) = \tan ti + \frac{t}{t^2-1}j + 3tk$, then $r(t)$ is continuous for all t with $t \neq 1, -1$ and $\frac{2n\pi+\pi}{2}$ where $n \in Z$.

4.3.2 Derivatives and Integrations

Definition 6 *The* derivative of the vector valued function $r(t)$ *is the vector function formed by differentiating each components of* $r(t)$ *with respect to* t*. That is* $r'(t) = (f'(t), g'(t), h'(t))$

Example: Let $r(t) = sinti + e^{2t}j + ln(3t - 1)k$, then $r'(t) = costi + 2e^{2t}j + \frac{3}{3t-1}$ and $r'(0) = i + 2j - 3k$, $r'(1) = cos1i + 2e^2j + \frac{3}{2}k$
Example: Let $r(t) = \frac{1}{2t}i + t^3 + 5k$, then $r'(t) = -\frac{1}{2t^2}i + 3t^2j$ and $r'(1) = -\frac{1}{2}i + 3j$, $r'(-2) = -\frac{1}{8}i + 12j$.

Definition 7 *If* $r(t) = (f(t), g(t), h(t))$*, then the integral of* $r(t)$ *is given by*

$$\int r(t)dt = \left(\int f(t)dt, \int g(t)dt, \int h(t)dt\right)$$

Example: Let $r(t) = sinti + e^{2t}j + (3t^2 - 1)k$, then

$$\int_0^{\pi} r(t)dt = 2i + \frac{1}{2}(e^{2\pi} - 1)j + (\pi^3 - \pi)k$$

Example: Let $r(t) = ti + (te^t)j + (3t^2)k$, then

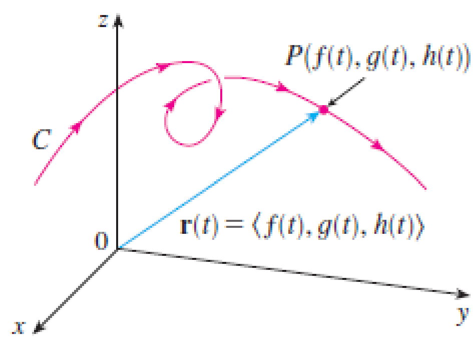
$$\int r(t)dt = (\frac{1}{2}t^2 + c_1)i + (te^t + e^t + c_2)j + (t^3 + c_3)k$$

4.4 Curves, Arc Length and Tangent vectors

4.4.1 Curves and Parametric Representations

Definition 8 *A space curve (or simply curve) is the range of a continuous vector-valued function on an interval of real numbers.*

Example: Line, circle, parabola, ellipse,circular helix, etc are same examples of curves.



Suppose C be a smooth curve in space, then the coordinates (x, y, z) can be represented by the equation

$$\boxed{x = f(t),\ y = g(t) \qquad \text{and} \qquad z = h(t)}$$

is called **parametric representation** of the curve C and t is called **parameter**. A vector equation of this curve is given by

$$r(t) = f(t)i + g(t)j + h(t)k$$

Parametric representation of Line

The parametric representation of line l which passing through the point(x_o, y_o, z_o) and parallel to the vector $v = ai + bj + ck$ is given by

$$x = x_o + at, y = y_o + bt \quad and \quad z = z_o + ct \quad where \quad t \in \Re$$

and the vector equation is

$$r(t) = (x_o + at)i + (y_o + bt)j + (z_o + ct)k \quad t \in \Re$$

EXAMPLE: Find the parametric representation of the line that passing through $(2, -3, 2)$ and parallel to the vector $v = 2i + 4j - 3k$.
Solution: Let $(x_o, y_o, z_o) = (2, -3, 2)$ and the parallel vector $v = ai + bj + ck = 2i + 4j - 3k$, then the parametric representation of this line is

$$x = x_o + at = 2 + 2t, y = y_o + bt = -3 + 4t, z = z_o + ct = 2 - 3t$$

for $t \in \Re$.

$$\Rightarrow r(t) = (2 + 2t)i + (-3 + 4t)j + (2 - 3t)k \quad for \quad t \in \Re$$

EXAMPLE: Find the parametric representation of the line segment from the point $(2, -1, 3)$ to $(3, 4, 0)$?
Solution: Let $p(x_o, y_o, z_o) = (2, -1, 3)$ and $q(x_1, y_1, z_1) = (3, 4, 0)$, then the parallel vector to the line is $v = \overrightarrow{pq} = i + 5j - 3k$.

Thus, the parametric representation of this line is

$$x = x_o + at,\ y = y_o + bt \text{ and } z = z_o + ct$$
$$\Rightarrow x = 2 + t,\ y = -1 + 5t \text{ and } z = 3 - 3t \text{ where } 0 \leq t \leq 1$$

Remark: The parametric representation of the line segment from the point (x_o, y_o, z_o) to (x_1, y_1, z_1) is given by:

$$x = x_o + (x_1 - x_o)t, y = y_o + (y_1 - y_o)t, z = z_o + (z_1 - z_o)t \quad for \quad 0 \leq t \leq 1$$

Parametric representation of Circles and Ellipse

Definition 9 *The parametric representation of circle center at (a, b) and radius r, $(x - a)^2 + (y - b)^2 = r^2$, is*

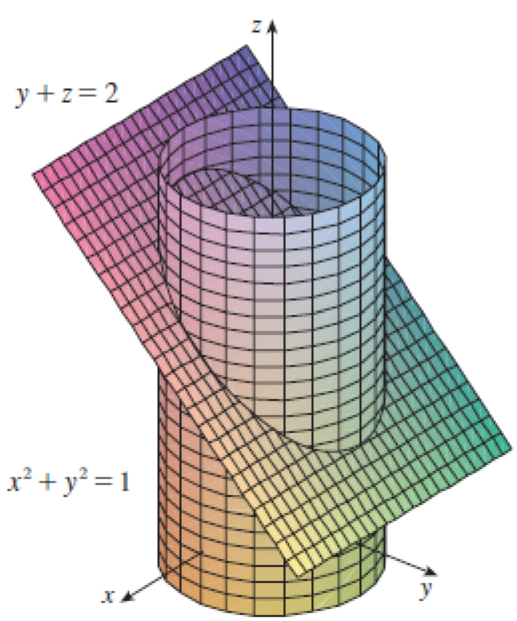
$$r(t) = (a + rcost)i + (b + rsint)j \quad for \quad 0 \leq t \leq 2\pi$$

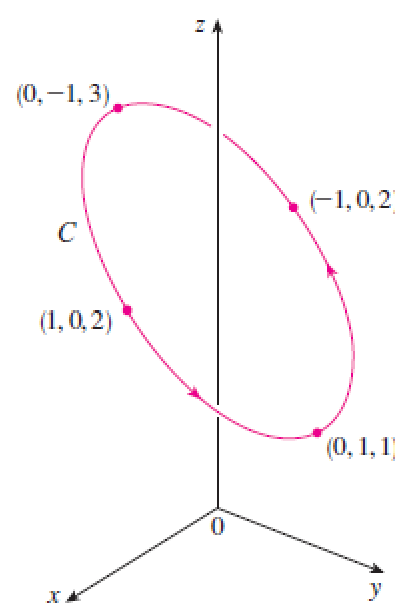
Definition 10 *The parametric representation of ellipse, $\frac{(x-x_o)^2}{a^2} + \frac{(y-y_o)^2}{b^2} = 1$, is*

$$r(t) = (x_o + acost)i + (y_o + bsint)j \quad for \quad 0 \leq t \leq 2\pi$$

Examples: Find the parametric(vector) representation of the following curves of

1. circle center at the origin and radius 2 in the first quadrant
2. circle center at $(3, -2$ and radius 4.
3. an ellipse $\frac{(x+3)^2}{4} + \frac{(y-1)^2}{9} = 1$
4. an ellipse $4x^2 + 9y^2 = 36$
5. intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.
   Solution5: show how the plane and the cylinder intersect, which shows the curve of intersection C, which is an ellipse.





The projection of C onto the xy-plane is the circle $x^2 + y^2 = 1, z = 0$. So we know

$$x = cost, y = sint \quad for \quad 0 \le t \le 2\pi$$

$$\rightarrow z = 2 - y = 2 - sint$$

So we can write parametric equations for C as

$$x = \cos t, y = \sin t, = 2 - \sin t \quad 0 \le t \le 2\pi$$

The corresponding vector equation is

$$r(t) = \cos ti + \sin tj + (2 - \sin t)k \qquad 0 \le t \le 2\pi$$

Other Solutions: Exercise
Examples: Describe the curve defined by the vector function

1. $r(t) = (1 + t, 2 + 5t, -1 + 6t)$
   Solution: The corresponding parametric equations are

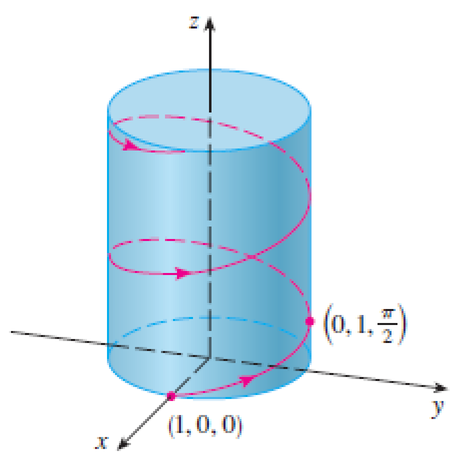
   $$x = 1 + t, y = 2 + 5t, z = -1 + 6t$$

   which is the parametric equations of a line passing through the point $(1, 2, -1$ and parallel to the vector $v = i + 5j + 6k$.

2. $r(t) = costi + sintj + tk$
   Solution: The corresponding parametric equations are

   $$x = cost, y = sint, z = t$$

   since $x^2 + y^2 = (cost)^2 + (sint)^2 = 1$, the curve must lie on the circular cylinder $x^2 + y^2 = 1$. The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$ in the xy-plane. Since $z = t$, the curve spirals upward around the cylinder as t increases. Thus, the curve is called a **circular helix**.



3. $r(t) = 3costi + 3sintj$ for $0 \le t \le \pi$

4. $r(t) = costi - sintj$ for $0 \le t \le 2\pi$

5. $r(t) = (2cost, 3sint, 0)$

6. $r(t) = (6 - t^2)i + \frac{1}{2}tj$ for $-2 \le t \le 4$

Velocity, Speed and Acceleration

If $r(t) = f(t)i + g(t)j + h(t)k$ is represent the object, then

* The **position** is $r(t) = f(t)i + g(t)j + h(t)k$

  The **Velocity** is $v(t) = r'(t) = f'(t)i + g'(t)j + h'(t)k$

* The **Speed** is $\|v(t)\| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$

* Acceleration is $a(t) = r''(t) = f''(t)i + g''(t)j + h''(t)k$

Example: Find the velocity, speed and acceleration of an object
1) $r(t) = e^t sinti + e^t costj + e^t k$ 2) $r(t) = coshti + sinhtj + tk$
Example: Find the position, velocity and speed of an object having acceleration
1) $a(t) = -\cos ti - \sin tj$; $v_o = k$ and $r_o = i$
2 $a(t) = e^t i + e^{-t} j$; $v_o = i - j + \sqrt{2}k$ and $r_o = i + j$

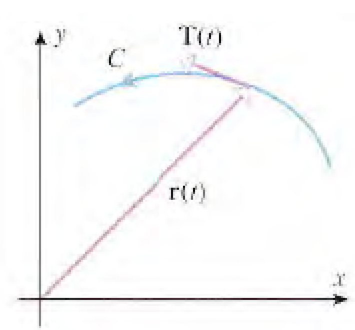
4.4.2 Tangent Vector

Definition 11 *If C is the graph of a smooth vector-valued function $r(t)$ in 2-space or 3-space, then the vector $r'(t)$ is nonzero,* ***tangent*** *to C at any point t and $r''(t)$ is* ***normal*** *to C at any point t. Thus, by normalizing $r'(t)$ we obtain a unit vector*

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

We call $T(t)$ the ***unit tangent vector*** *to C at t. And the unit* ***normal vector*** *is given by*

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{r''(t)}{\|r''(t)\|}$$



Example:
a) Find the unit tangent vector to the graph of $r(t) = t^2 i + t^3 j$ at $t = 2$.
Solution: Since

$$r'(t) = 2ti + 3t^2 j$$

is the tangent vector at any point t. Thus, the unit tangent vector at $t = 2$ is

$$T(2) = \frac{r'(2)}{\|r'(2)\|} = \frac{4i + 12j}{\sqrt{16 + 144}} = \frac{1}{\sqrt{10}}i + \frac{3}{\sqrt{10}}$$

b) Find the unit tangent and unit normal vectors to the ellipse $x^2 + 4y^2 = 4$ at $(\sqrt{2}, \frac{1}{\sqrt{2}})$.

4.4.3 Arc Length of the curve

Definition 12 *Let C be a curve with a piecewise smooth parametrization r defined on $[a, b]$. Then the length L of C is defined by*

$$L = \int_a^b \|r'(t)\| dt = \int_a^b \left\| \frac{dr}{dt} \right\| dt$$

Note: If $r(t) = x(t)i + y(t)j + z(t)k$ for $a \leq t \leq b$, then

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example: Find the length L of the segment of the circular helix

$$r(t) = costi + sintj + tk \quad for \quad 0 \leq t \leq 2\pi$$

Solution: since $r'(t) = -sinti + costj + k$ and

$$\|r'(t)\| = \sqrt{(-cost)^2 + sin^2t + 1} = \sqrt{2}$$

Thus, the arc length is

$$L = \int_0^{2\pi} \|r'(t)\| dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi$$

Example: Find the length L of the twisted cubic curve

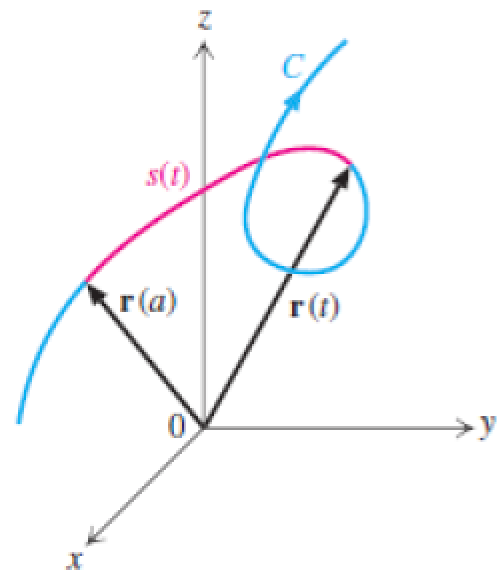
$$r(t) = ti + \frac{\sqrt{6}}{2}t^2 j + t^3 k \quad for \quad -1 \leq t \leq 1$$

Example: Find the length the length of the curve

$$r(t) = (1 + 3t^2, 4 + 2t^3) \quad for \quad 0 \leq t \leq 1$$

Example: Show that the circumference of the circle center at the origin and radius r is $2\pi r$.
Example: Find the length of circle center at the origin and radius 4 in the first quadrant.

4.4.4 Arc Length Function



Let C be a smooth curve parameterized on the interval I by

$$r(t) = x(t)i + y(t)j + z(t)k, fort \in I$$

and let a be a fixed number in I. We define **the arc length function s** by

$$s(t) = \int_a^t \|r'(u)\|du = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2}du$$

Notice that if $t \geq a$, then $s(t)$ is the length of the portion of the curve between $r(a)$ and $r(b)$, and if $r(t)$ denotes the position of an object at time $t \geq a$, then $s(t)$ is the distance traveled by the object between time a and time t.Or equivalently,

$$\frac{ds}{dt} = \|\frac{dr}{dt}\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

$$\Rightarrow s = s(t) = \int_a^t \|r'(u)\|du$$

It is often useful to parameterize a curve with respect to arc length because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system. If a curve $r(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function given by above Equation, then we may be able to solve for t as a function of s:$t = s(t)$. Then the curve can be re-parameterized in terms of s by substituting for t:$r = r(t(s))$.

Example: Suppose that $r(t) = ti + t^2j + t^3k$, then find the arc length function.

Solution:since $r'(t) = i + 2tj + 3t^2k$, then

$$\frac{ds}{dt} = \|\frac{dr}{dt}\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{1 + 4t^2 + 9t^4}$$

Example: Re-parameterize the helix $r(t) = costi + sintj + tk$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

Solution: The initial point $(1, 0, 0)$ corresponds to the parameter value $t = 0$.

$$\frac{ds}{dt} = \|r'(t)\| = \sqrt{2}$$

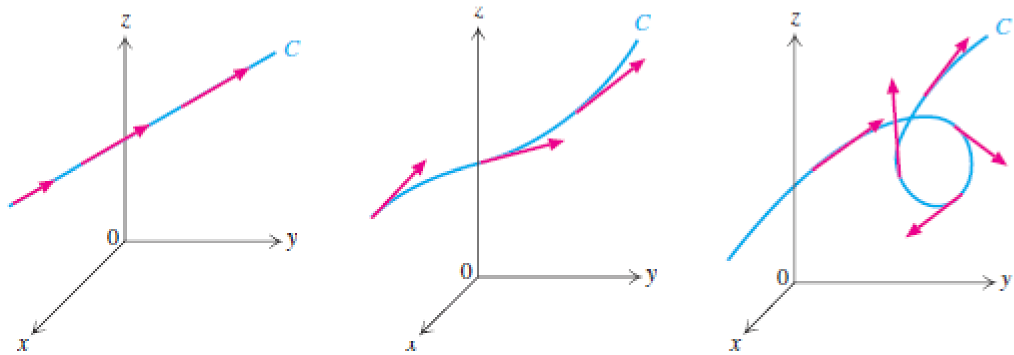
and the arc length function is

$$s = s(t) = \int_0^t \|r'(u)\|du = \int_0^t \sqrt{2}du = \sqrt{2}t$$

Therefore, $t = \frac{s}{\sqrt{2}}$ and the required re-parametrization is obtained by substituting for t:

$$r(t(s)) = cos(\frac{s}{\sqrt{2}})i + sin(\frac{s}{\sqrt{2}})j + \frac{s}{\sqrt{2}}k$$

4.4.5 Curvature



Suppose that C is the graph of a smooth vector valued function that is parameterized in terms of arc length function. The "sharpness" of the bend in C is closely related to $\frac{dT}{ds}$, which is the rate of change of the unit tangent vector T with respect to s.

Definition 13 *Let C have a smooth parametrization r such that r' is differentiable. Then the curvature κ of C is defined by the formula*

$$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|}$$

Examples:

- Find the Curvature of a straight line.
- Find the Curvature for a circle of radius r.
  Solution: We can take the circle to have center the origin, and then a parametrization is

$$r(t) = rcosti + rsintj$$

  $\Rightarrow r'(t) = -rsinti + rcostj$ and $\|r'(t)\| = r$
  so, $T(t) = \frac{r'(t)}{\|r'(t)\|}$
  and $T'(t) = -costi - sintj$ and $\|T'(t)\| = 1$
  So that the Curvature is

$$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{1}{r}$$

- Find the Curvature of $r(t) = (2sint, 2cost, 4t)$

Theorem: If $r(t)$ is a smooth vector valued function, then for each value of t at which $T'(t)$ and $r''(t)$ exist, the Curvature κ can be expressed as

$$\kappa(t) = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3}$$

4.5 Divergence and Curl

Definition 14 *Let $F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$ be a differentiable vector field function. Then*

- *the divergence of F, denoted by **DivF**, is given by*

$$DivF = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

  which is a scalar field function.

- *the **Curl** of F is given by*

$$CurlF = (\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z})i + (\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x})j + (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})k$$

  which is a vector field function.

Note: $curlF = \nabla xF = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$

Example: Find the Divergence and Curl of the following vector field functions
a) $F(x,y,z) = (2xy, xyz^2 - sin(yz), ze^{x+y})$
Solution:i) Divergence of F is

$$DivF = \frac{\partial(2xy)}{\partial x} + \frac{\partial(xyz^2 - sin(yz)}{\partial y} + \frac{\partial(ze^{x+y})}{\partial z}$$

$$DivF = 2y + xz^2 - zcos(yz) + e^{x+y}$$

$$DivF(1,1,0) = 2 + e^2$$

ii) Curl of F is

$$CurlF = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & xyz^2 - sin(yz) & ze^{x+y} \end{vmatrix}$$

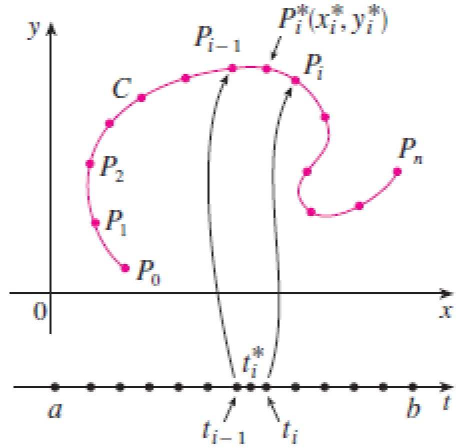
$$CurlF = (ze^{x+y} - 2xyz + ycos(yz))i - ze^{x+y}j + (yz^2 - 2x)k$$

$$CurlF(1,1,0) = -i - 2k$$

b) $F(x,y,z) = (y, 2xz, ze^x)$
c) $F(x,y) = (x^3 + y)i + (2x^2y + y^3)j$

4.6 Line Integrals

4.6.1 Line Integral of Scalar functions



Suppose that a plane curve C given by the parametric equations

$$r(t) = x(t)i + y(t)j \quad a \leq t \leq b$$

is smooth curve.If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$. If $\triangle s_i$ represents the arc length of C_i and the norm of the partition $\|p\|$ to be the maximum of the arc lengths $\triangle s_i$, then the **Line integral** of the scalar function $f(x, y)$ on the curve C:$r(t) = x(t)i + y(t)j$ for $a \leq t \leq b$ is

$$\int_C f(x,y)ds = \lim_{\|p\| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\triangle s_i$$

provided the limit exists.

Definition 15 *If f is defined on a smooth curve C: $r(t) = x(t)i + y(t)j$ for $a \leq t \leq b$, then the* ***line integral*** *of f along C is*

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt = \int_a^b f(x(t), y(t))\|r'(t)\|dt$$

Note: The line integral of $f(x, y, z)$ over the curve C:$r(t) = x(t)i + y(t)j + z(t)k$ for $a \leq t \leq b$ is given by

$$\int_C f(x,y,z)ds = \int_a^b f(x(t), y(t), z(t))\|r'(t)\|dt$$

Examples: Evaluate the line integral of f over the curve C

- $f(x, y, z) = xy$,over C:$r(t) = (4cost, 4sint, -3)$ for $0 \leq t \leq \frac{\pi}{2}$
  Solution: Since $x(t) = 4cost, y(t) = 4sint, z(t) = -3$ and

$$r'(t) = (-4sint, 4cost, 0) \qquad and \qquad \|r'(t)\| = 4$$

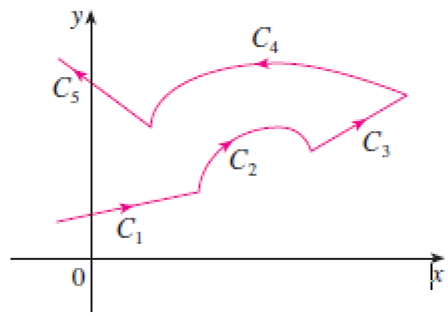
  Thus, the line integral of f over C is

$$\int_C xyds = \int_0^{\frac{\pi}{2}} x(t)y(t)\|r'(t)\|dt = \int_0^{\frac{\pi}{2}} (4cost)(4sint)4dt$$

$$= \int_0^{\frac{\pi}{2}} 64costsint \qquad let \qquad u = sint, du = costdt$$

$$= \int_0^{\frac{\pi}{2}} 64udu = 32u^2 = 32sin^2t|_0^{\frac{\pi}{2}} = 32$$

- $f(x, y) = 2 + x^2y$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.
- $f(x, y, z) = ysinz$, where C is a circular helix given by $r(t) = costi + sintj + tk, 0 \leq t \leq 2\pi$



Suppose C is a **piecewise-smooth curve**; that is, $C = C_1 \cup C_2 \cup ... \cup C_n$ where $C_1, C_2, ..., C_n$ are smooth curves.Then

$$\int_C fds = \int_{C_1} fds + \int_{C_2} fds + ... + \int_{C_n} fds$$

Example: Evaluate $\int_C 2xds$, where C is the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment from $(1, 1)$ to $(1, 2)$.
Solution: Since C is piecewise smooth and $C = C_1 \cup C_2$. The Parametric representation of those curves are The parametric of the parabola: let $x = t$ and then $y = t^2$,

$$C_1 : r_1(t) = ti + t^2j, 0 \leq t \leq 1$$

The parametric of the line is

$$C_2 : r_2(t) = (1 + (1 - 1)t)i + (1 + (2 - 1)t)j = i + (1 + t)j, 0 \leq t \leq 1$$

Thus, the line integral is

$$\int_C 2xds = \int_{C_1} 2xds + \int_{C_2} 2xds$$
$$= \int_0^1 2x(t)\|r_1'(t)\|dt + \int_0^1 2x(t)\|r_2'(t)\|$$
$$= \int_0^1 2t\sqrt{1 + 4t^2}dt + \int_0^1 2dt$$
$$= \frac{5\sqrt{5} - 1}{6} + 2$$

4.6.2 Line Integral of Vector Field Functions

Definition 16 *Let F be a continuous vector field defined on a smooth curve C given by a vector function $r(t)$,$a \leq t \leq b$. Then the **line integral** of F along C is*

$$\int_C F.dr = \int_a^b F(r(t)).r'(t)dt$$

Note: Since $dr = (dx, dy, dz)$, $F = (f, g, h)$ and $r' = \frac{dr}{dt}$, then

$$\int_C F.dr = \int_C (fdx + gdy + hdz) = \int_a^b (fx' + gy' + hz')dt$$

Example: Find the line integral, $\int_C F.dr$, where
i) $F(x, y, z) = (z, x, y)$ over the curve C:$r(t) = (cost, sint, 3t)$ for $0 \leq t \leq 2\pi$
Solution: Since $x = cost, y = sint$ and $z = 3t$, we have

$$F(r(t)) = (3t, cost, sint) = 3ti + costj + sintk$$

$$and \qquad r'(t) = -sinti + costj + 3k$$

Therefore, the line integral over C is

$$\int_C F.dr = \int_0^{2\pi} F(r(t)).r'(t)dt = \int_0^{2\pi} (-3tsint + cos^2t + 3sint)dt$$

$$= (3tcost - 3sint + \frac{1}{2}t - \frac{1}{4}sin2t - 3cost)|_0^{2\pi}$$
$$= 7\pi$$

ii) $F(x, y) = (-y, -xy)$ over the curve C is a circular arc $x^2 + y^2 = 1$ in the first quadrant.
Solution: The parametric representation of C is

$$r(t) = costi + sintj, 0 \leq t \leq \frac{\pi}{2}$$

and $F(r(t)) = -sinti - costsintj$, $r'(t) = -sinti + cost$
Thus, the line integral is

$$\int_C F.dr = \int_0^{\frac{\pi}{2}} (sin^2t - cos^2tsint)dt$$
$$= \frac{1}{2}t - \frac{1}{4}sin2t + \frac{1}{3}cos^3t)|_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{4} - \frac{1}{3}$$

Ex: Suppose an object moves along the parabola $y = x^2$ from the point $(-1, 1)$ to $(2, 4)$. find the total work done if the motion is caused by the force field $F(x, y) = (x^2 + y^2, 3x^2y)$.
Remark:1) If $C = C_1 \cup C_2 \cup ... \cup C_n$, then

$$\int_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr + ... + \int_{C_n} F.dr$$

2) $\int_{-C} F.dr = -\int_C F.dr$
Example: Let C be the curve consisting of the quarter circle $x^2 + y^2 = 1$ in xy-plane from $(1, 0)$ to $(0, 1)$, followed by the horizontal line segment from $(0, 1)$ to $(2, 1)$. Compute $\int_C (x^2ydx + y^2dy)$
Solution: Since C is piecewise smooth and $C = C_1 \cup C_2$, where C_1 is circle and C_2 is horizontal line segment. The parametric of those curves are

$$C_1 : r_1(t) = costi + sintj, 0 \leq t \leq \frac{\pi}{2}$$

$$C_2 : r_2(t) = 2ti + j, 0 \leq t \leq 1$$

The line integral over C is

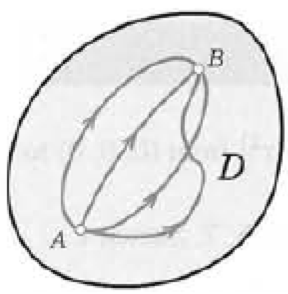
$$\begin{aligned}\int_C F.dr &= \int_{C_1} F.dr + \int_{C_2} F.dr \\ &= \int_0^{\frac{\pi}{2}} F(r(t)).\|r_1'(t)\|dt + \int_0^1 F(r(t)).\|r_2'(t)\|dt \\ &= -\int_0^{\frac{\pi}{2}} cos^2tsin^2tdt + \int_0^{\frac{\pi}{2}} sin^2tcostdt + \int_0^1 8t^2dt \\ &= -\int_0^{\frac{\pi}{2}} (costsint)^2dt + \frac{1}{3}sin^3t|_0^{\frac{\pi}{2}} + +\frac{8}{3}t^3|_0^1 \\ &= -\int_0^{\frac{\pi}{2}} (\frac{1}{2}sin2t)^2dt + \frac{1}{3} + \frac{8}{3} \\ &= -\int_0^{\frac{\pi}{2}} \frac{1}{4}(\frac{1}{2} - \frac{1}{2}cos4t)dt + 3 \\ &= -\frac{\pi}{16} + 3\end{aligned}$$

Example: Evaluate $\int_C F.dr$ where $F = (x, -z, 2y)$ from $(0,0,0)$ straight to $(1,1,0)$, then to $(1,1,1)$, back to $(0,0,0)$.

4.6.3 INDEPENDENCE OF PATH AND CONSERVATIVE VECTOR FIELDS

Definition 17 ***(Independent of Path)***

The line integral $\int_C F(r).dr$ is **independent of path** on a set D if for any point A and B in D, the line integral has the same value over any paths in D having initial point A and terminal point B. And the vector field F is **Conservative** on D.



Definition: If F is a field defined on D and $F = \nabla f$ for some scalar function f on D, then f is called a **Potential function** for F.
Theorem:(Path independence)
A line integral $\int_C F(r).dr$ with continuous F_1, F_2, F_3 in D is **path independent** in D iff $F = (F_1, F_2, F_3)$ is the gradient of some function f in D, that is

$$F = \nabla f, or F_1 = \frac{\partial f}{\partial x}, F_2 = \frac{\partial f}{\partial y}, F_3 = \frac{\partial f}{\partial z}$$

Theorem:(Fundamental theorem for line integral)
If the vector field F is conservative and $F = \nabla f$ in D from the point A to B, then the line integral is

$$\int_C F(r).dr = f(B) - f(A)$$

Theorem: (Test for a Conservative Field)
The vector field $F = (F_1, F_2, F_3)$ in the line integral $\int_C F(r).dr$ is Conservative in D iff

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}, or \qquad CurlF = 0$$

Note: If $F = (F_1, F_2)$ is conservative in D iff $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$
Examples: Show that the line integral $\int_C F(r).dr$ is path independent in any domain D and find its value where C run from the point A to B.
a)$F = (e^x cosy, -e^x siny)$; from $A = (0, 0)$ to $B = (2, \frac{\pi}{4})$
Solution: Since $F_1 = e^x cosy$ and $F_2 = -e^x siny$, and so

$$\frac{\partial F_1}{\partial y} = -e^x siny = \frac{\partial F_2}{\partial x}$$

Thus, the vector field F is conservative, so that the line integral $\int_C F(r).dr$ is independent of path. Let f be the potential function of F such that $F = \nabla f$.

$$\Rightarrow \frac{\partial f}{\partial x} = F_1 = e^x cosy, \frac{\partial f}{\partial y} = F_2 = -e^x siny$$

Take $\frac{\partial f}{\partial x} = F_1 = e^x cosy$, integrate w.r.t x, we get

$$f(x, y) = e^x cosy + h(y)$$

Differentiate f w.r.t y, then we get

$$\frac{\partial f}{\partial y} = -e^x siny + h'(y) = -e^x siny$$

$$\Rightarrow h'(y) = 0$$

$$\Rightarrow h(y) = c$$

Therefore, $f(x, y) = e^x cosy + c$ is the potential function of F. Thus, the line integral from A to B is

$$\int_C F.dr = \int_A^B F.dr = f(B) - f(A) = f(2, \frac{\pi}{4}) - f(0, 0) = \frac{e^2\sqrt{2}}{2} - 1$$

b) $F = (2xyz^2, x^2z^2 + zcos(yz), 2x^2yz + ycos(yz))$ from $A = (0, 0, 1)$ to $(1, \frac{\pi}{4}, 2)$.
Solution: Since the $CurlF$ is

$$CurlF = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + zcos(yz) & 2x^2yz + ycos(yz) \end{vmatrix}$$

$$\Rightarrow CurlF = (2x^2z + cosyz - yzsinyz - 2x^2z - cosyz + yzsinyz)i - (4xyz - 4xyz)j + (2xz^2 - 2xz^2)k = 0$$

Thus, the vector field F is conservative and $\int_C F(r).dr$ is path independent. So, there is a scalar function f such that $F = \nabla f$.

$$\frac{\partial f}{\partial x} = 2xyz^2, \frac{\partial f}{\partial y} = x^2z^2 + zcos(yz), \frac{\partial f}{\partial z} = 2x^2yz + ycos(yz)$$

Take $\frac{\partial f}{\partial x} = 2xyz^2$, integrate w.r.t x and we get

$$f(x, y, z) = x^2yz^2 + h(y, z)$$

Differentiate w.r.t y and we get

$$\frac{\partial f}{\partial y} = x^2z^2 + \frac{\partial h}{\partial y} = x^2z^2 + zcos(yz)$$

$$\Rightarrow \frac{\partial h}{\partial y} = zcos(yz)$$

$$\Rightarrow h(y,z) = sinyz + g(z)$$

$$\Rightarrow f(x,y,z) = x^2yz^2 + sin(yz) + g(z)$$

Differentiate f w.r.t z and

$$\frac{\partial f}{\partial z} = 2x^2yz + ycos(yz) + g'(z) = 2x^2yz + ycos(yz)$$

$$\Rightarrow g'(z) = 0 \Rightarrow g(z) = c$$

$\therefore f(x,y,z) = x^2yz^2 + sin(yz) + c$ is the potential function of F. Thus the line integral using fundamental theorem of line integral is

$$\int_C F.dr = \int_A^B F.dr = f(B) - f(A) = \pi + 1$$

c) $F = (y^2, 2xy + e^{3z}, 3ye^{3z})$ from $A = (0,1,0)$ to $B = (4,2,1)$.
d) $F = (z^2 + 2xy, x^2, 3xz)$ from $A = (2,1,3)$ to $B = (4,-1,0)$.

Note: If F is conservative and the curve C is a simple closed path in D, then $\int_C F(r).dr = 0$

4.7 Green's Theorem

The curve in the plane defined parametrically by

$$C = \{(x, y) : r(t) = x(t)i + y(t)j, a \leq t \leq b\}$$

C is **closed** if its two end points are the same,i.e $(x(a), y(a)) = (x(b), y(b))$. A curve C is **simple** if it does not intersect itself, except at the end point.

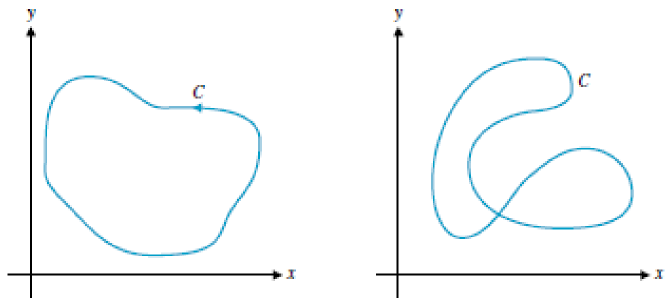


Figure 4.2: 1. simple closed curve and 2. closed but not simple curve.

Theorem:(Green's Theorem) Let C be a simple closed positively oriented path in the plane. Let D be the region enclosed by C, together with C. Let $F_1(x, y), F_2(x, y), \frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ be continuous on D. Then

$$\int_C F.dr = \int\int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dA$$

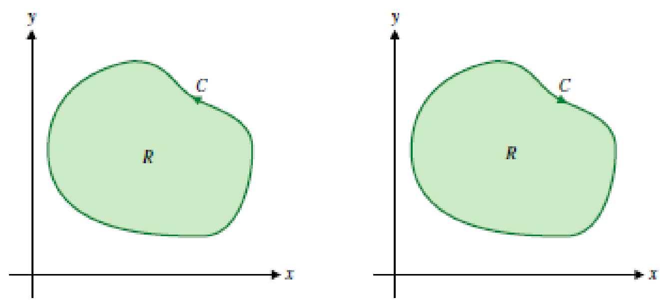
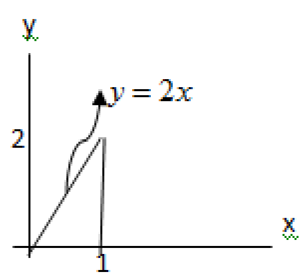


Figure 4.3: counter clockwise(positive) and clockwise(negative) directions

Example: Use Green's Theorem to evaluate the following line integrals

1. $\int_C (x^2 ydx + xdy)$, where C is a triangle whose vertices $(0, 0), (1, 0)$ and $(1, 2)$ oriented CCD.
   Solution Since

$$F_1 = x^2 y, F_2 = x \Rightarrow \frac{\partial F_1}{\partial y} = x^2, \frac{\partial F_2}{\partial x} = 1$$



Therefore,

$$\begin{aligned}\int_C (x^2ydx + xdy) &= \int\int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dA \\ &= \int\int_D (1-x^2)dA \\ &= \int_0^1 \int_0^{2x} (1-x^2)dydx \\ &= \int_0^1 (2x - 2x^3)dx \\ &= (x^2 - \frac{1}{2}x^4)|_0^1 \\ &= \frac{1}{2}\end{aligned}$$

2. $\int_C ((e^x - y^3)dx + (cosy + x^3)dy)$, where C is the int circle $x^2+y^2=1$ in the ccd.
Solution: Since $\frac{\partial F_1}{\partial y} = -3y^2$ and $\frac{\partial F_2}{\partial x} = 3x^2$. Thus, the line integral over C by using Green's Theorem is

$$\begin{aligned}\int_C F.dr &= \int\int_D ((\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dA \\ &= \int\int_D (3x^2+3y^2)dA \\ &= 3\int\int_D (x^2+y^2)dA\end{aligned}$$

Using polar coordinates, $x = rcos\theta, y = rsin\theta$ for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$

$$\begin{aligned}\int_C F.dr &= 3\int_0^{2\pi}\int_0^1 r^3 drd\theta \\ &= 3\int_0^{2\pi} \frac{1}{4}d\theta \\ &= \frac{3}{4}\theta|_0^{2\pi} = \frac{3\pi}{2}\end{aligned}$$

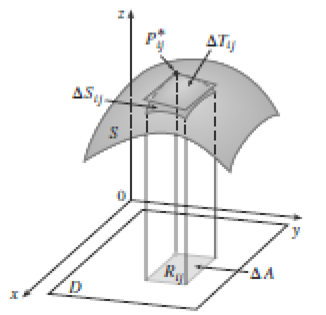
3. $\int_C (x^2ydx + (y+xy^2)dy)$, where C is the boundary of the region enclosed by $y = x^2$ and $x = y^2$.

4. $\int_C ((7y - e^{sinx})dx + (15x - sin(y^3+8y))dy)$, where C is the circle of radius 3 centered at $(5,-7)$.

5. $\int_C ((e^x + 6xy)dx + (8x^2 + siny^2)dy)$, where C is positively oriented boundary of the region bounded by the circles of radii 1 and 3, center at the origin and lying in the first quadrant.

4.8 Surface Integrals

4.8.1 Surface Integral of Scalar functions



Suppose f is a function of three variables whose domain includes a surface S. We divide S into patches S_{ij} with area $\triangle S_{ij}$. We evaluate f at a point P_{ij}^* in each patch, multiply by the area $\triangle S_{ij}$, and form the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*\triangle S_{ij}$$

Then we take the limit as the patch size approaches 0 and define the surface integral of f over the surface S as

$$\int\int_S f(x,y,z)dS = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})^*\triangle S_{ij}$$

If the surface S is a graph of a function of two variables, then it has an equation of the form $z = g(x,y), (x,y) \in D$ We first assume that the parameter domain D is a rectangle and we divide it into smaller rectangles R_{ij} of equal size. The patch S_{ij} lies directly above the rectangle R_{ij} and the point P_{ij}^* in S_{ij} is of the form $(x_i^*, y_j^*, g(x_i^*, y_j^*))$, then

$$\triangle S_{ij} \cong \triangle T_{ij} = \sqrt{(g_x(x_i,y_j))^2 + (g_y(x_i,y_j))^2 + 1}\triangle A$$

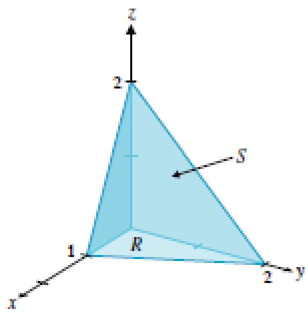
If f is continuous on S and g has continuous derivatives, then

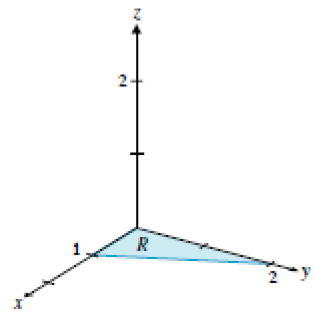
$$\int\int_S f(x,y,z)dS = \int\int_D f(x,y,g(x,y))\sqrt{(g_x(x,y))^2 + (g_y(x,y))^2 + 1}dA$$

Definition 18 *If the surface S is given by $z = g(x,y)$ for (x,y) in the region $D \subset \Re^2$, where g has continuous first partial derivatives, then*

$$\int\int_S f(x,y,z)dS = \int\int_D f(x,y,g(x,y))\sqrt{(g_x(x,y))^2 + (g_y(x,y))^2 + 1}dA$$

Example: Evaluate $\int\int_S 3zdS$, where the surface S is the portion of the plane $2x + y + z = 2$ lying in the first octant.
Solution: we have $z = g(x,y) = 2 - 2x - y$, then $g_x(x,y) = -2, g_y(x,y) = -1$





$$D = (x, y) : 0 \leq x \leq 1, 0 \leq y \leq 2 - 2x$$

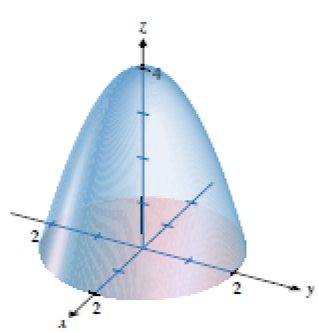
thus, the surface integral is

$$\begin{aligned}\int\int_S 3z dS &= \int\int_S 3(2 - 2x - y) dS \\ &= \int_0^1 \int_0^{2-2x} 3(2 - 2x - y)\sqrt{(g_x)^2 + (g_y)^2 + 1} dy dx \\ &= \int_0^1 3(2 - 6x + 2x^2)\sqrt{6} dx \\ &= 3\sqrt{6}(2x - 3x^2 + \frac{2}{3}x^3|_0^1 \\ &= \frac{-3\sqrt{6}}{3}\end{aligned}$$

Example: Evaluate $\int\int_S z dS$, where the surface S is the portion of the paraboloid $z = 4 - x^2 - y^2$ lying above the xy-plane.

Solution: Substituting $z = 4 - x^2 - y^2$, we have

$$\int\int_S z dS = \int\int_S (4 - x^2 - y^2) dS$$



$$D = \{(x, y) : x^2 + y^2 = 4\}$$

This gives

$$\int\int_S z dS = \int\int_D (4 - x^2 - y^2)\sqrt{(z_x)^2 + (z_y)^2 + 1} dA = \int\int_D (4 - x^2 - y^2)\sqrt{4x^2 + 4y^2 +} dA$$

Using polar coordinate system, $x = r cos\theta, y = r sin\theta$ for $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. Then, we have

$$\begin{aligned}\int\int_S &= \int_0^{2\pi}\int_0^2 (4 - r^2)\sqrt{4r^2 + 1} r dr d\theta \\ &= \frac{280\pi\sqrt{17} - 41\pi}{60}\end{aligned}$$

Example: Evaluate $\int_S (3x^2+3y^2+3z^2)dS$, where S is the sphere $x^2+y^2+z^2=4$.
Solution: We divide the sphere into two parts:
S_1 is $z=\sqrt{4-x^2-y^2}$ above the xy-plane.
S_2 is $z=-\sqrt{4-x^2-y^2}$ below the xy-plane.
Then $S=S_1\cup S_2$ and the surface integral is

$$\int\int_S (3x^2+3y^2+3z^2)dS = \int\int_{S_1}(3x^2+3y^2+3z^2)dS+\int\int_{S_2}(3x^2+3y^2+3z^2)dS$$

Since $D=\{(x,y): x^2+y^2=4\}$, then

$$\begin{aligned}\int\int_S (3x^2+3y^2+3z^2)dS &= 2\int\int_{S_1}(3x^2+3y^2+3z^2)dS\\ &= 2\int\int_D (12)\sqrt{(z_x)^2+(z_y)^2+1}dA\\ &= 2\int_0^{2\pi}\int_0^2 (12)\frac{2}{\sqrt{4-r^2}}rdrd\theta\\ &= 24\int_0^{2\pi}(-2\sqrt{4-r^2})|_0^2 d\theta\\ &= 96\theta|_0^{2\pi}\\ &= 192\pi\end{aligned}$$

Exercise:

1. Evaluate $\int\int_S ydS$, where S is the surface $z=x+y^2$ for $0\le x\le 1, 0\le y\le 2$.
   Ans: $\frac{13\sqrt{2}}{3}$
2. Evaluate $\int\int_S y^2z^2dS$, where S is the surface $z=\sqrt{x^2+y^2}$ that lies between the planes $z=1$ and $z=2$. Ans: $\frac{21\pi}{\sqrt{2}}$
3. Evaluate $\int\int_S \sqrt{x^2+y^2+z^2}dS$, over the portion of cone $z=\sqrt{x^2+y^2}$ below the plane $z=1$.
4. Evaluate $\int\int_S \sqrt{x+y+z}dS$, over the portion of plane $x+y=1$ in the first octant for which $0\le z\le 1$.
   Ans: $\frac{3\sqrt{2}}{2}$
5. Evaluate $\int\int_S x^2+y^2dS$, where S is composed of the part of the paraboloid $z=\sqrt{1-x^2-y^2}$ above the xy-plane, and the part of the xy-plane that lies inside the circle $x^2+y^2=1$

4.9 Surface Integral over Oriented Surface(Flux Integral)

Definition 19 *Let F(x,y,z) be a continuous vector field defined on an oriented surface S with unit normal vector n. The surface integral of F over S (or the flux of F over S) is given by*

$$\int\int_S F.dS = \int\int_S F.ndS$$

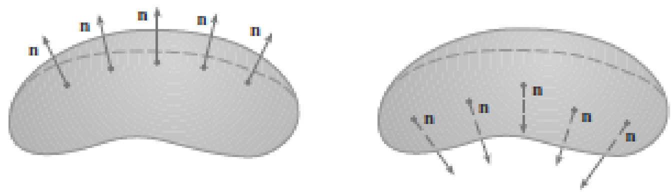


Figure 4.4: oriented upward and downward normal vectors

For a surface $z = f(x, y)$ oriented upward and D is the projection of S in xy-plane, then

$$n = \frac{-f_x i - f_y j + k}{\sqrt{(f_x)^2 + (f_y)^2 + 1}}$$

and $dS = \sqrt{(f_x)^2 + (f_y)^2 + 1}dA$
Thus, the flux integral is given by

$$\int\int_S F.dS = \int\int_S F.ndS = \int\int_D F.(-f_x i - f_y + k)dA$$

For a surface $z = f(x, y)$ oriented dawn ward and D is the projection of S in xy-plane, then

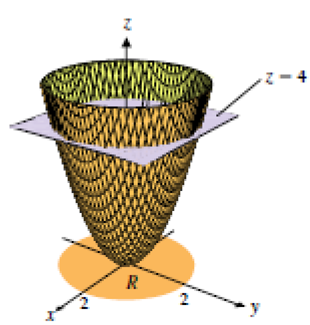
$$n = \frac{f_x i + f_y j - k}{\sqrt{(f_x)^2 + (f_y)^2 + 1}}$$

and $dS = \sqrt{(f_x)^2 + (f_y)^2 + 1}dA$
Thus, the flux integral is

$$\int\int_S F.dS = \int\int_S F.ndS = \int\int_D F.(f_x i + f_y - k)dA$$

1. Compute the flux of the vector field $F(x, y, z) = (x, y, 0)$ over the portion of the paraboloid $z = x^2 + y^2$ below $z = 4$ (oriented with upward-pointing normal vectors).
   Solution: Since the surface S is oriented upward and the projection of S in the xy-plane is

$$D = \{(x, y) : x^2 + y^2 = 4\}$$



Thus, the flux is

$$\begin{aligned}\int\int_S F.ndS &= \int\int_D F.(-z_x i - z_y j + k)dA \\ &= \int\int_D (xi + yj).(-2xi - 2yj + k)dA \\ &= \int\int_D (-2x^2 - 2y^2)dA \\ &= \int_0^{2\pi}\int_0^2 (-2r^3)drd\theta \\ &= -6\pi\end{aligned}$$

2. Let S be the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy-plane and suppose S is oriented up. Find the flux of the vector field $F(x, y, z) = xi + yj + zk$ over S.
Solution: Since the surface S oriented up and the projection region is

$$D = \{(x, y) : x^2 + y^2 = 1\}$$

Thus, the flux integral is

$$\begin{aligned}\int\int_S F.dS &= \int\int_D F.(-z_x i - z_y j + k)dA \\ &= \int\int_D (x, y, 1 - x^2 - y^2).(2x, 2y, 1)dA \\ &= \int\int_D (x^2 + y^2 + 1)dA \\ &= \int_0^{2\pi}\int_0^1 (r^2 + 1)rdrd\theta \\ &= \frac{3\pi}{2}\end{aligned}$$

3. Compute the flux of the vector field $F(x, -1, 2x^2) = (x, y, 0)$ over the surface $z = x^2 + y^2$ above the region in the xy-plane bounded by the parabolas $x = 1 - y^2$ and $x = y^2 - 1$ directed downward.

4. Compute the flux of the vector field $F(x, -1, 2x^2) = (x + y, y + z, x + z)$ over the portion of the plane $x + y + z = 1$ in the first octant, oriented by unit normals with positive components.

Note: If $S = S_1 \cup S_2 \cup ... \cup S_n$, then

$$\int\int_S F.dS = \int\int_{S_1} F.dS + \int\int_{S_2} +... + \int\int_{S_n} dS$$

Example: Let $F(x, y, z) = zk$ and S be the unit sphere $x^2 + y^2 + z^2 = 1$, oriented with the normal that is directed outward. Compute $\int\int_S F.dS$
Solution: Since S have two parts;upper hemisphere S_1 and lower hemisphereS_2.
on S_1, $z = \sqrt{1 - x^2 - y^2}$ and on S_2, $z = -\sqrt{1 - x^2 - y^2}$ and the projection of S_1 and S_2 are

$$D_1 = D_2 = \{(x, y) : x^2 + y^2 = 1\}$$

$$\begin{aligned}
\therefore \int\int_S F.dS &= \int_{S_1} F.dS + \int\int_{S_2} F.dS \\
&= \int\int_{D_1} F.(-z_x i - z_y j + k)dA + \int\int_{D_2} F.(z_x i + z_y j - k)dA \\
&= 2\int\int_D (0,0,\sqrt{1-x^2-y^2}).(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1)dA \\
&= 2\int\int_D \sqrt{1-x^2-y^2}dA \\
&= \frac{4\pi}{3}
\end{aligned}$$

4.10 Stokes' Theorem

Theorem:Stokes' Theorem
Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let F be a vector field whose components have continuous partial derivatives on an open region in $\Re^3$ that contains S. Then

$$\int_C F.dr = \int\int_S CurlF.dS$$

EXAMPLE:Evaluate $\int_C F.dr$, where $F = (x, y, z) = -y^2 i + xj + z^2 k$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)
Solution: The curve C is an ellipse
We first compute

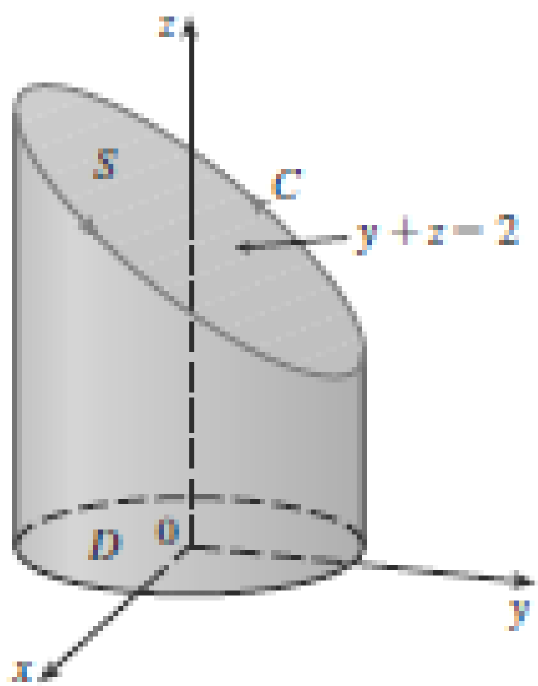


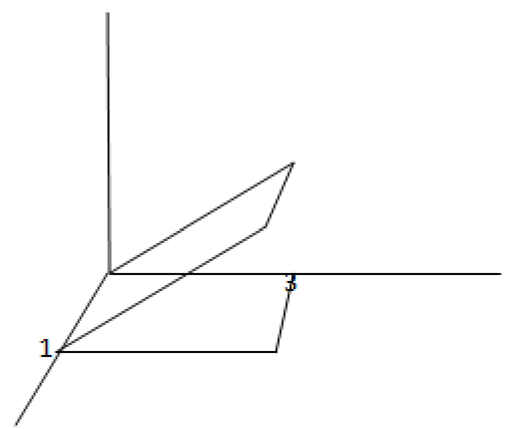
Figure 4.5: intersection surface

$$CurlF = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1+2y)k$$

Since $z = 2 - y$ and $D = |(x, y) : x^2 + y^2 \leq 1$. Then the line integral is

$$\begin{aligned} \int_C F.dr &= \int\int_S CurlF.dS \\ &= \int\int_D CurlF.(-z_x i - z_y j + k)dA \\ &= \int\int_D (1+2y)k.(j+k)dA \\ &= \int_0^{2\pi}\int_0^1 (1+2rsin\theta)rdrd\theta \\ &= \pi \end{aligned}$$

Exercises: Find the work performed by the vector field $F = x^2 i + 4xy^3 j + xy^2 k$ on a particle that traverses the rectangular C in the plane $z = y$ with CCW direction.
Solution:since $CurlF = 2xyi - y^2 j + 4y^3 k$ and the plane surface S enclosed by C is assigned a downward orientation to make the orientation of C positive.



Therefore, the work done is

$$\begin{aligned} W = \int_C F.dr &= \int\int_S CurlF.ndS \\ &= \int\int_D CurlF.(z_x i + z_y j - k)dA \\ &= \int\int_D (2xyi - y^2 j + 4y^3 k)(0i + 0j - k)dA \\ &= \int\int_D (-y^2 - 4y^3)dA \\ &= -\int_0^1\int_0^3 (y^2 + 4y^3)dydx \\ &= -90 \end{aligned}$$

Exercises: Evaluate $\int_C F.dr$ using stoke's Theorem

1. $F = z^2 i + 2xj - y^3 k$, C is the circle $x^2 + y^2 = 1$ in the xy-plane with CCW orientation looking down the positive z-axis.

2. $F = (z + sinx)i + (x + y^2)j + (y + e^z)k$, C is the intersection of sphere $x^2 + y^2 + z^2 = 1$ and cone $z = \sqrt{x^2 + y^2}$ with CCW orientation down the positive z-axis.

4.11 THE DIVERGENCE THEOREM

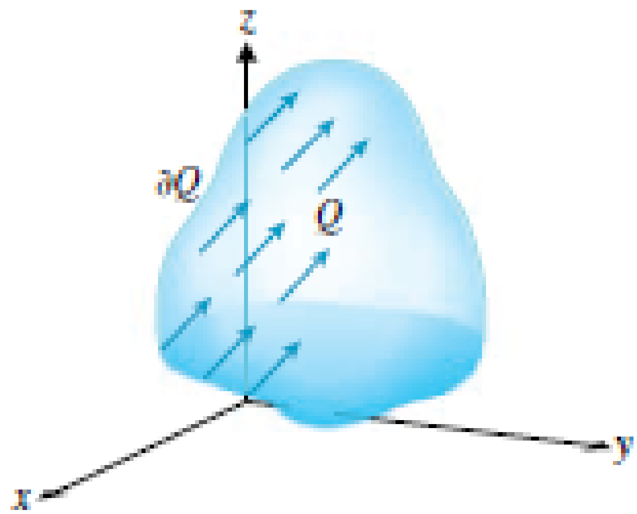


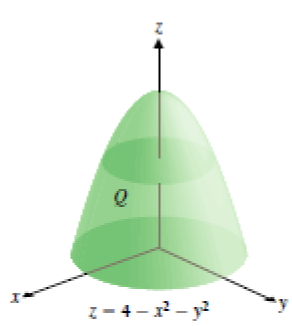
Figure 4.6: Flow of fluid across $\partial Q = S$

Theorem:(Divergence Theorem)
Suppose that $Q \Subset \Re^3$ is bounded by the closed oriented surface S and that $n(x, y, z)$ denotes the exterior unit normal vector to S. Then, if the components of $F(x, y, z)$ have continuous first partial derivatives in Q, then the flux is given by

$$\int\int_S F.ndS = \int\int\int_Q DivFdV$$

Example: Let Q be the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Find the flux of the vector field $F(x, y, z) = (x^3, y^3, z^3)$ over the surface S.
Solution: The divergence of F is

$$DivF(x, y, z) = \nabla.F(x, y, z) = 3x^2 + 3y^2 + 3z^2$$



By Divergence Theorem, the flux of F over S is given by:

$$\begin{aligned}\int\int_S F.ndS &= \int\int\int_Q DivFdV = \int\int\int_Q (3x^2+3y^2+3z^2)dV \\ &= 3\int_0^{2\pi}\int_0^2\int_0^{4-r^2}(r^2+z^2)rdzdrd\theta \\ &= 3\int_0^{2\pi}\int_0^2 (r^3(4-r^2)+\frac{1}{3}(4-r^2)^3r)drd\theta \\ &= 96\pi\end{aligned}$$

Exercises: Use the divergence Theorem to evaluate the out ward flux of the following vector fields

1. $F = zk$ across the sphere $x^2+y^2+z^2=4$
   Solution: Since $DivF(x,y,z)=1$. Therefore, the flux is

   $$\int\int_S F.ndS = \int\int\int_Q DivFdV = \int\int\int_Q dV$$

   Using spherical coordinates, (ρ,ϕ,θ),

   $$x=\rho sin\phi cos\theta, y=\rho sin\phi sin\theta, z=\rho cos\phi, 0\leq\theta\leq 2\pi; 0\leq\phi\leq\pi; 0\leq\rho\leq 2$$

   $$\begin{aligned}\therefore\int\int_S F.ndS &= \int_0^{2\pi}\int_0^{\pi}\int_0^2 \rho^2 sin\phi d\rho d\phi d\theta \\ &= -\frac{8}{3}cos\phi|_0^{\pi}d\theta \\ &= \frac{16}{3}\theta|_0^{2\pi} \\ &= \frac{32\pi}{3}\end{aligned}$$

2. $F = 3xi+4yj+5zk$, S is the sphere $x^2+y^2+z^2=9$
3. $F = x^3i+y^3j+z^3$, and Q be the region bounded by the xy-plane and the hemi-sphere $x^2+y^2+z^2=4$
4. $F = x^3i+y^3j+z^2$, and Q be the region bounded by the circular cylinder $x^2+y^2=9$ and the plane $z=0$ and $z=2$.